A Fire Fighters’ Problem
joint work with Rolf Klein and Christos Levcopoulos

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- Construction of firebreaks
- Simple theoretical model
- Expanding circle (unit speed)
- Firefighter speed: $v \in [1, \infty)$
- Barrier construction outside the fire
Motion Planning and Firefighting

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- Speed \( v \approx 4 \)
- Starting close to the fire
- Barrier blocks extension
- Enclose the fire? Speed?

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Motion Planning and Firefighting

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- Speed $v \approx 4$
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Extremes on speed $v$
Simple bounds, FollowFire strategy

\[
\text{FireFighterSpeed} \quad v \in (1, 2\pi + 1)
\]

FollowFire Strategy

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Simple bounds, FollowFire strategy

**FireFighterSpeed**

\[ \nu \in (1, 2\pi + 1) ? \]

**FollowFire Strategy**

- Start on the boundary
Simple bounds, FollowFire strategy

\[ \nu \in (1, 2\pi + 1) \] ?

FollowFire Strategy

- Start on the boundary
- Allowed angle?

\[ A = \alpha x \cos(\alpha) \]
Simple bounds, FollowFire strategy

FireFighterSpeed

\[ v \in (1, 2\pi + 1) ? \]

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\[ v \in (1, 2\pi + 1) \]

**FollowFire Strategy**

- Start on the boundary
- Allowed angle?
- *Riding* the fire
Simple bounds, FollowFire strategy

**FireFighterSpeed**

\[ v \in (1, 2\pi + 1) \]

**FollowFire Strategy**

- Start on the boundary
- Allowed angle?
- *Riding* the fire
- Log. Spiral around \( Z \)
- Excentricity \( \alpha \)
  \[
  \cos(\alpha) = \frac{1}{v}
  \]
FollowFire Strategy for $\nu = 5.27$!

Logarithmic spiral of excentricity $\alpha$ around $Z$ ($\frac{1}{\nu} = \cos(\alpha)$)!
FollowFire Strategy for $v = 5.27!$

Logarithmic spiral of excentricity $\alpha$ around $p_0 \left( \frac{1}{v} = \cos(\alpha) \right)$. 
FollowFire Strategy for $v = 5.27!$

Excentricity $\alpha$ around wrapping center $Z_1$ ($\frac{1}{v} = \cos(\alpha)$)!
FollowFire: Free String Wrapping!

- $v = 5.27$ ($\alpha = 1.38$)
- $\log(p_0, p_1), \log(p_1, p_2)$
- Free string: $F_1(l)$:
  - Wrapping around $\log(p_0, p_1)$
FollowFire: Free String Wrapping!

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- $\log(p_0, p_1), \log(p_1, p_2)$
- Free string: $F_1(l)$: Wrapping around $\log(p_0, p_1)$

- $\nu = 3.07$ ($\alpha = 1.24$)
- Wrapping around $\log(p_1, p_2)$
FollowFire: Free String Wrapping!

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- $\log(p_0, p_1), \log(p_1, p_2)$
- Free string: $F_1(l)$: Wrapping around $\log(p_0, p_1)$
- $\nu = 3.07$ ($\alpha = 1.24$)
- Wrapping around $\log(p_1, p_2)$
  
  Wrapping around wrappings!

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Experimental approach!
FollowFire: Successful?

\[ v = 2.69 \ (\alpha = 1.19): \]
8 rounds!

\[ v = 2.593 \ (\alpha = 1.175): \]
Simulation did not succeed!

**Successful for which** \( v \in (1, \infty) ? \)

**Lower and upper bounds on** \( v \)! **Proofs!**
Theorem

- To enclose the fire a spiralling strategy requires speed \( v_c > \frac{1+\sqrt{5}}{2} \approx 1.618 \)
- The FollowFire strategy is successful if \( v > v_c \approx 2.6144 \)
- As \( v \) decreases to \( v_c \), the number of rounds tends to \( \infty \)

Proof Sketch!
Spiralling strategies!
Visit four axes in cyclic order
Visit axes in increasing distance

Theorem
Each “spiralling” strategy must have speed $v > 1.618$ (golden ratio) to be successful.
Spiralling strategies!
- Visit four axes in cyclic order
- Visit axes in increasing distance

**Theorem**

*Each “spiralling” strategy must have speed $v > 1.618\ldots$ (golden ratio) to be successful.*
Proof of lower speed bound: suppose $v \leq 1.618$.

By induction:
On reaching $p_i$, interval of length $A$ below $p_{i-1}$ is on fire.
Proof of lower speed bound: suppose $v \leq 1.618$.

Inductive Step:
After arriving $p_{i+1}$ fire moves at least $x + A$.
Proof of lower speed bound: suppose $v \leq 1.618$.

Inductive Step:

After arriving $p_{i+1}$, fire moves at least $x + A$. 

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Proof of lower speed bound: suppose $v \leq 1.618$.

On reaching $p_i+1$:
1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

\[ \Rightarrow \frac{1}{v(v-1)}x + \frac{1}{v-1}A \leq \frac{y}{v} \]
\[ \Rightarrow x + A \leq \frac{y}{v} \]

from $v^2 - v \leq 1$
Theorem

- FollowFire strategy is successful if $v > v_c \approx 2.6144$
Theorem

- FollowFire strategy is successful if \( v > v_c \approx 2.6144 \)

Sketch of a sketch! When gets the free string to zero?

1. Parameterize free strings for coil \( j \) (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim’s Theorem and Cauchy’s Residue Theorem
FollowFire Wrapping process!

Free strings $F_j/\phi_j$ parameterized by length of starting spirals!

$|\text{Log}(p_0, p_1)| = l_1$

$|\text{Log}(p_0, p_1)| + |\text{Log}(p_1, p_2)| = l_2$

$F_j: l \in [0, l_1]$

$\phi_j: l \in [l_1, l_2]$
FollowFire Drawing backwards tagents!

Free strings $F_j/\phi_j$ parameterized by lenght of starting spirals!
FollowFire Drawing backwards tagents!

Free strings $F_j/\phi_j$ parameterized by lenght of starting spirals!
2. Linkage: Structural Properties

Parameterized by length $l$ of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!
Parameterized by length $l$ of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!

**Lemma**

\[ L_{j-1} + F_j = \cos \alpha L_j \]

**Lemma**

\[ \frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}} \]
3. Recursive linear differential equations

\[ F_j'(l) - \frac{\cos \alpha}{F_0(l)} F_j(l) = -\frac{F_{j-1}(l)}{F_0(l)} \]
3. Recursive linear differential equations

\[ \phi_j'(l) - \frac{\cos \alpha}{\phi_0(l)} \phi_j(l) = -\frac{\phi_{j-1}(l)}{\phi_0(l)} \]
3. Recursive linear differential equations

Interaction: $F_{j+1}(l_1) = \phi_{j+1}(l_1)$ and $F_{j+1}(0) = \phi_j(l_2)$

Crosswise initial values for DEQ

$$F_{j+1}(l) = F_0(l) \left( \frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)} \, dt \right)$$

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Will the free string crash onto the previous coil?

Will the free strings be negative for some \( j \)?
3. Recursive linear differential equations

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Will the free string crash onto the previous coil?

Will the free strings be negative for some $j$?

Inserting discrete values $l_1$ and $l_2$ (end of intervals) is sufficient!
Iterated substitutions results in the following sums!

\[ F_j(l_1) = \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{2\pi}{\sin \alpha} \right)^\nu \phi_{j-1-\nu}(l_2) \]

\[ \phi_j(l_2) = \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left( \frac{\alpha}{\sin \alpha} \right)^\nu \hat{F}_{j-\nu}(l_1) \]
4. End of intervals: Crosswise recursions

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\]

Will one of these values be negative for some \( j \)?

For which \( \alpha \)?
Classical trick for recursions: Coefficients of power series!

\[ F(X) := \sum_{j=0}^{\infty} F_j X^j \quad \text{and} \quad \phi(X) := \sum_{j=0}^{\infty} \phi_j X^j \]

where \( F_j := F_j(l_1) \) and \( \phi_j := \phi_j(l_2) \)
6. and 7. Formal power series equation

\[ \sum_{j=0}^{\infty} F_j X^j = F(X) = \frac{e^{qX} - rX}{e^{wX} - sX} \]

where \( q, r, w, s \) functions of \( \alpha \) and \( \alpha = \arccos \left( \frac{1}{v} \right) \)
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where \( q, r, w, s \) functions of \( \alpha \) and \( \alpha = \arccos \left( \frac{1}{v} \right) \)

- Sign of coefficients \( F_j \)? Classical approach fails!
- Ph. Flajolet: Analytic combinatorics
- Singularities of the right hand side function!
- Denominator \( e^{wX} - sX \):
  No real singularities for \( \alpha > \alpha_c \approx 1.1783 \) \((v \approx 2.6144)\)
- Pringsheim’s Theorem: Some \( F_j \) is negative for \( \alpha > \alpha_c \)!
  Success and quantified version by Cauchy’s Residue Theorem!
Number of rounds as function of speed

$\dot{j}$

$V$

A Fire Fighters' Problem
Results and open problems

Theorem

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**Theorem**

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- Is $v > 2.6144$ the true lower bound?
Results and open problems

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- Given feasible speed, how to minimize area burned?
Results and open problems

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- Is $v > 2.6144$ the true lower bound?
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- ...or time to completion?
- Starting points away from the fire?
Single round for $v > 3.7788$
Different fire shapes

International Handbook on Forest Fire Protection
Food and Agriculture Organization of the United Nations

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A Fire Fighters’ Problem
More open problems

- Using existing fire breaks
- Containing and extinguishing fire
- Cooperating fire fighters
- Path planning in dynamic environments
More open problems

- using existing fire breaks
More open problems

- using existing fire breaks
- containing & extinguishing fire
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