A Fire Fighters' Problem

joint work with Rolf Klein and Christos Levcopoulos

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Motion Planning and Firefighting

- Construction of firebreaks
- Simple theoretical model
- Expanding circle (unit speed)
- Firefighter speed: $v \in [1, \infty)$
- Barrier construction outside the fire

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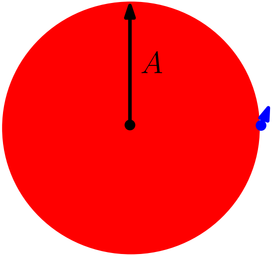

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- Starting close to the fire
- Barrier blocks extension
- Enclose the fire? Speed?

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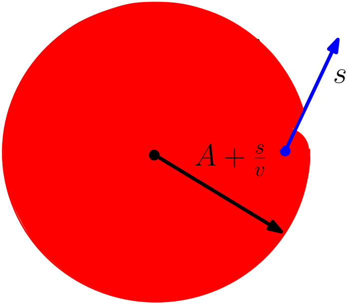

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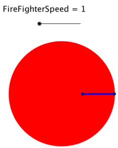
Extremes on speed v

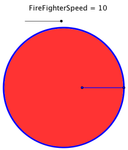
(Extremes!)

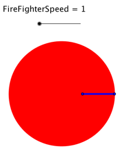


FireFighterSpeed

$v \in (1, 2\pi + 1)$ **?**

FollowFire Strategy





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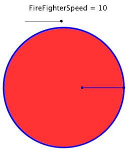
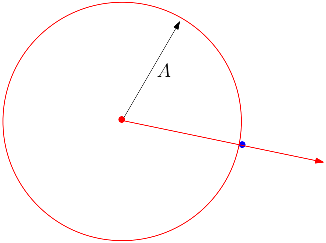
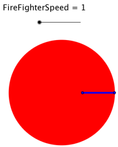




- Start on the boundary



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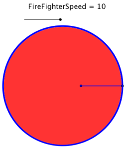
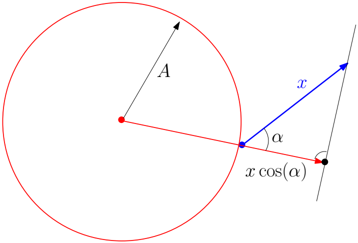
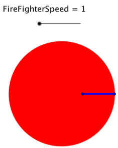




- Start on the boundary
- Allowed angle?

Simple bounds, FollowFire strategy



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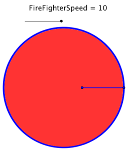

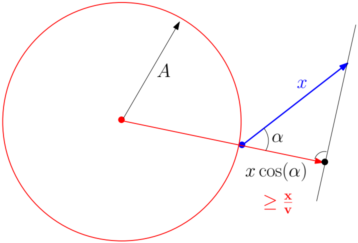

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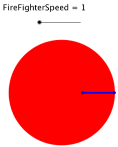

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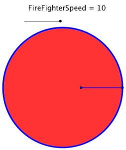

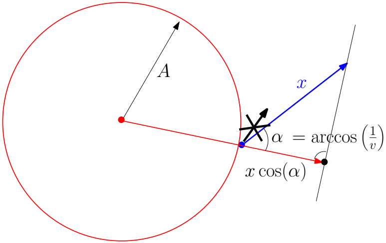

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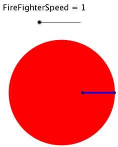

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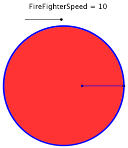
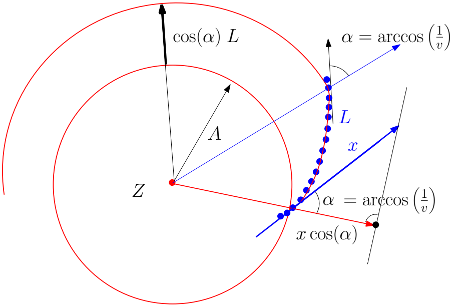
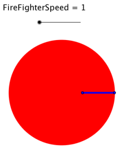




- Start on the boundary
- Allowed angle?
- *Riding* the fire



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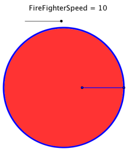
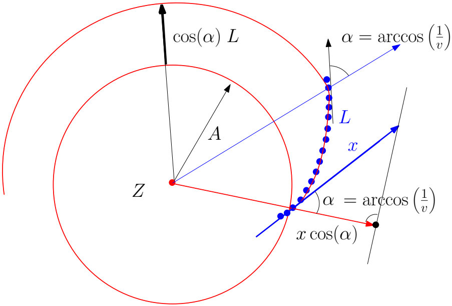




- Start on the boundary
- Allowed angle?
- *Riding* the fire
- Log. Spiral around Z
- Excentricity α
  $\cos(\alpha) = \frac{1}{v}$

FollowFire Strategy for $v = 5.27$!

Logarithmic spiral of excentricity α around Z ($\frac{1}{v} = \cos(\alpha)$)!

(First Part)

FollowFire Strategy for $v = 5.27$!

Logarithmic spiral of excentricity α around p_0 ($\frac{1}{v} = \cos(\alpha)$)!

(Second Part)

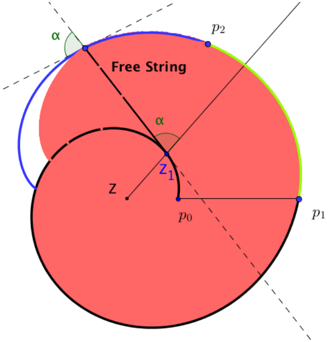
FollowFire Strategy for $v = 5.27$!

Excentricity α around wrapping center Z_1 ($\frac{1}{v} = \cos(\alpha)$)!

(Third part!)

FollowFire: Free String Wrapping!

- $v = 5.27$ $(\alpha = 1.38)$
- $\text{Log}(\mathbf{p_0}, \mathbf{p_1})$, $\text{Log}(\mathbf{p_1}, \mathbf{p_2})$
- Free string: $F_1(l)$:
  Wrapping around $\text{Log}(\mathbf{p_0}, \mathbf{p_1})$



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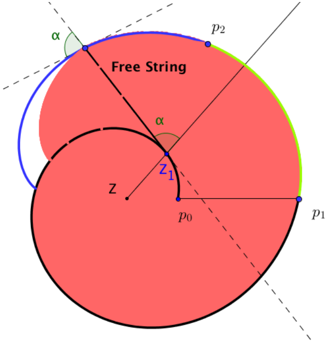
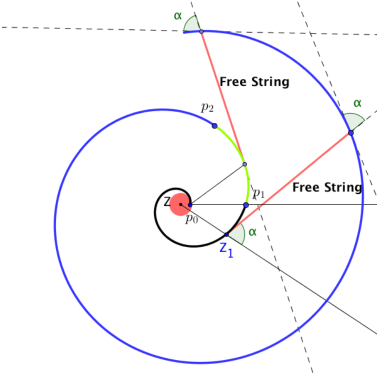


- $v = 3.07$ ($\alpha = 1.24$)
- Wrapping around $\mathrm{Log}(\mathbf{p_1}, \mathbf{p_2})$



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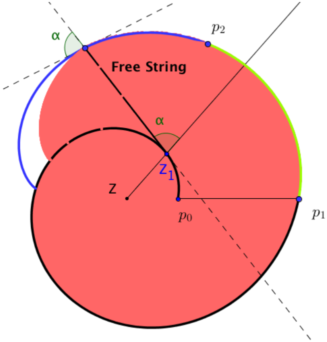

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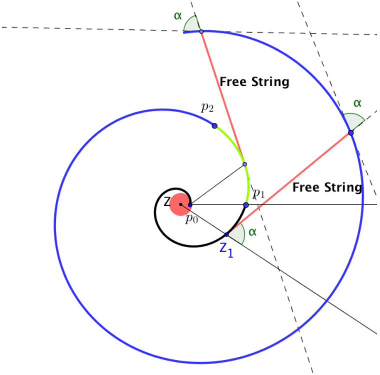
- Wrapping around $\mathrm{Log}(\mathbf{p_1}, \mathbf{p_2})$

Wrapping around wrappings!



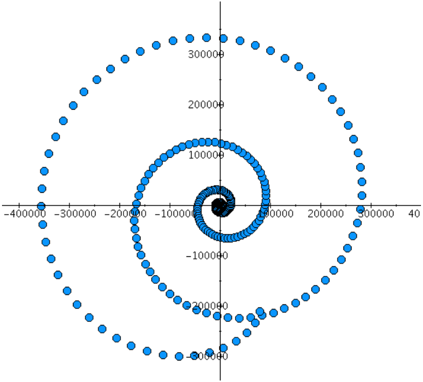
Experimental approach!

(Spiral Generator Appet!)

FollowFire: Successful?

$v = 2.69$ $(\alpha = 1.19)$:
8 rounds!

$v = 2.593$ $(\alpha = 1.175)$:
Simulation did not succeed!



Successful for which $v \in (1, \infty)$?

Lower and upper bounds on v! Proofs!

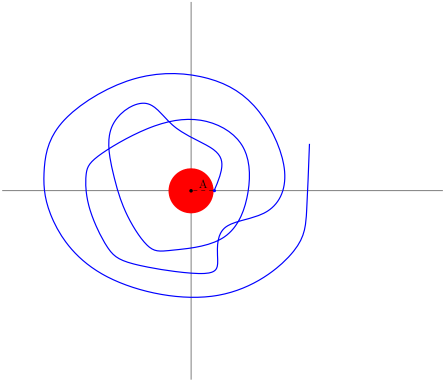
Theorem

- *To enclose the fire a spiralling strategy requires speed* $v_c > \frac{1+\sqrt{5}}{2} \approx 1.618$
- *The FollowFire strategy is successful if* $v > v_c \approx 2.6144$
- *As* v *decreases to* v_c*, the number of rounds tends to* ∞

Proof Sketch!

Lower bound construction, spiralling strategies!

- Spiralling strategies!
- Visit four axes in cyclic order
- Visit axes in increasing distance



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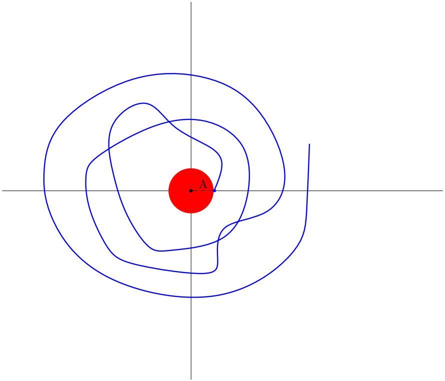
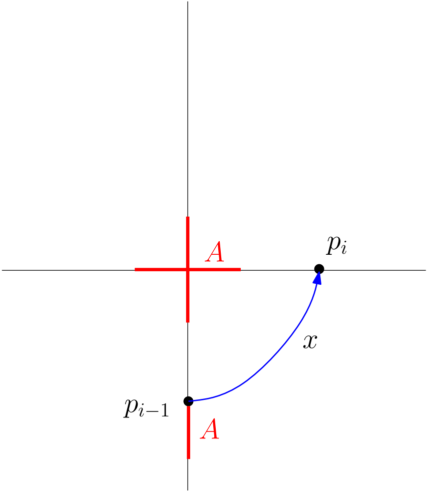


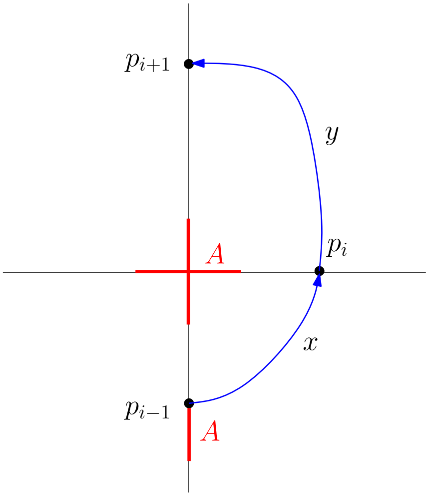
Theorem

Each "spiralling" strategy must have speed $v > 1.618\ldots$ (golden ratio) to be successful.

Proof of lower speed bound: suppose $v \leq 1.618$



By induction:
On reaching p_i,
interval of length A below
p_{i-1} is on fire



Inductive Step:

After arriving p_{i+1}
fire moves at least $x + A$

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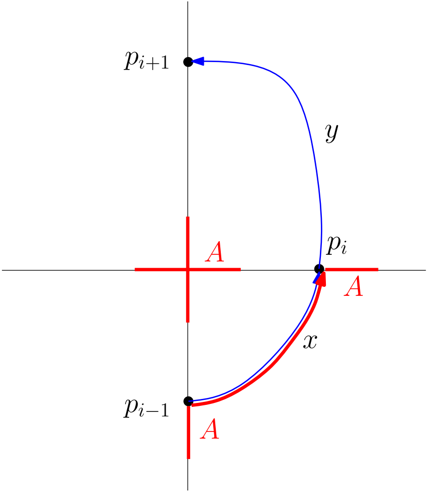

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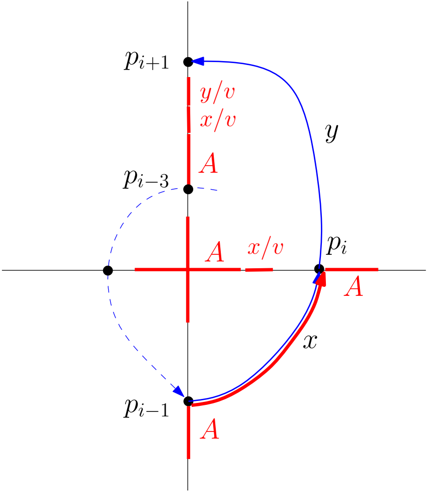


On reaching p_{i+1}:

1. $A + \frac{x}{v} \leq p_i \leq x$ and
2. $A + \frac{x}{v} + \frac{y}{v} \leq p_{i+1} \leq y$

$\Longrightarrow \frac{1}{v(v-1)}x + \frac{1}{v-1}A \leq \frac{y}{v}$

$\Longrightarrow x + A \leq \frac{y}{v}$

from $v^2 - v \leq 1$

Upper bound by FollowFire

Theorem

- *FollowFire strategy is successful if* $v > v_c \approx 2.6144$

Upper bound by FollowFire

Theorem

- *FollowFire strategy is successful if $v > v_c \approx 2.6144$*

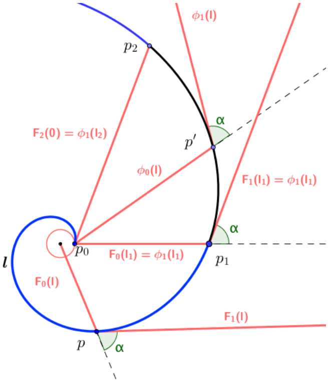
Sketch of a sketch! When gets the free string to zero?

1. Parameterize free strings for coil j (Linkage)
2. Structural properties
3. Successive interacting differential equations
4. Inserting end of parameter interval
5. Coefficients of power series
6. Ph. Flajolet: Singularities
7. Pringsheim's Theorem and Cauchy's Residue Theorem

Upper bound: 1. Parameterize the free string

FollowFire Wrapping process!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!

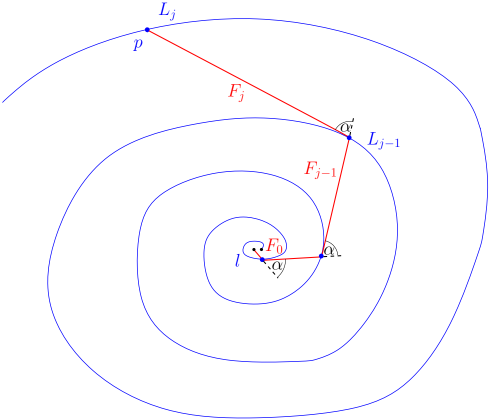


$$|Log(p_0, p_1)| = l_1$$
$$|Log(p_0, p_1)| + |Log(p_1, p_2)| = l_2$$

$$F_j\colon\ l \in [0, l_1]$$
$$\phi_j\colon\ l \in [l_1, l_2]$$

FollowFire Drawing backwards tagents!

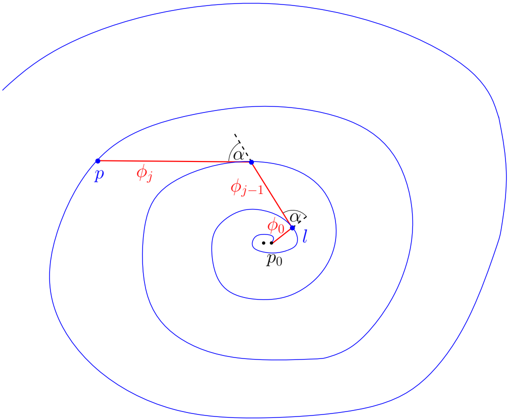
Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



Upper bound: 1. Parameterize the free string (Linkage)

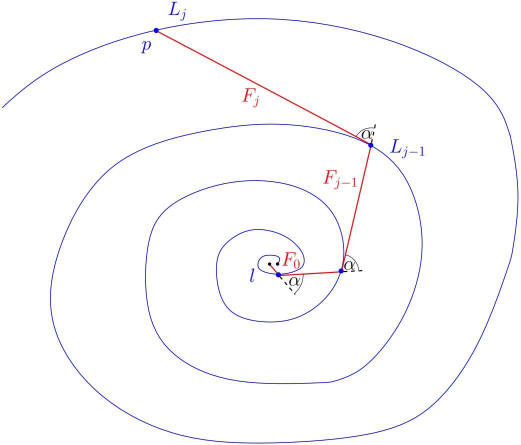
FollowFire Drawing backwards tagents!

Free strings F_j/ϕ_j parameterized by lenght of starting spirals!



Parameterized by lenght l of starting spirals!

$L_j(l)$ length of the curve! $F_j(l)$ (and $\phi_j(l)$) length of the free string!



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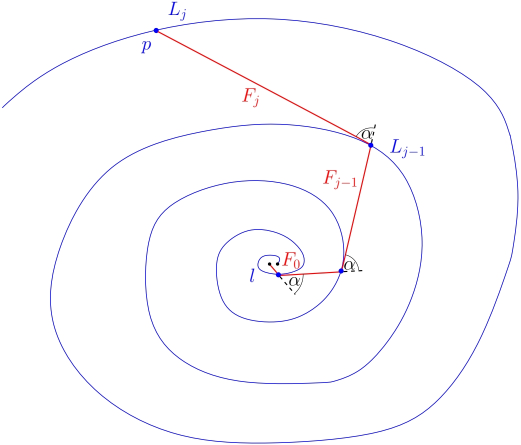
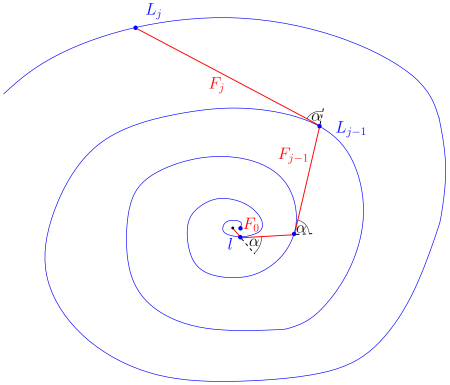


Lemma

$$L_{j-1} + F_j = \cos\alpha \, L_j$$

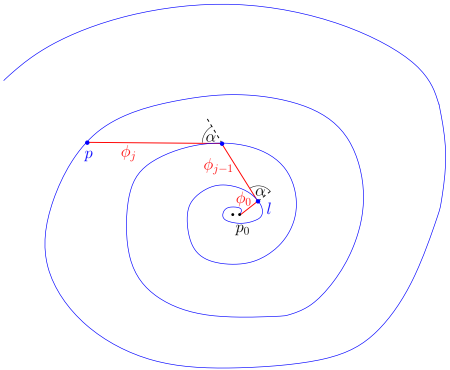
Lemma

$$\frac{L'_j}{L'_{j-1}} = \frac{F_j}{F_{j-1}}$$



$$F_j'(l) - \frac{\cos\alpha}{F_0(l)} F_j(l) = -\frac{F_{j-1}(l)}{F_0(l)}$$

3. Recursive linear differential equations



$$\phi_j'(l) - \frac{\cos\alpha}{\phi_0(l)}\phi_j(l) = -\frac{\phi_{j-1}(l)}{\phi_0(l)}$$

Interaction: $F_{j+1}(l_1) = \phi_{j+1}(l_1)$ and $F_{j+1}(0) = \phi_j(l_2)$

Crosswise initial values for DEQ

$$\begin{aligned} F_{j+1}(l) &= F_0(l)\left(\frac{\phi_j(l_2)}{F_0(0)} - \int_0^l \frac{F_j(t)}{F_0^2(t)}\,\mathrm{d}t\right) \\ \phi_{j+1}(l) &= \phi_0(l)\left(\frac{F_{j+1}(l_1)}{\phi_0(l_1)} - \int_{l_1}^l \frac{\phi_j(t)}{\phi_0^2(t)}\,\mathrm{d}t\right) \end{aligned}$$

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Will the free string crash onto the previous coil?

Will the free strings be negative for some j?

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Will the free string crash onto the previous coil?

Will the free strings be negative for some j?

Inserting discrete values l_1 and l_2 (end of intervals) is sufficient!

Iterated substitutions results in the following sums!

$$
\begin{aligned}
F_j(l_1) &= \frac{F_0(l_1)}{F_0(0)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{2\pi}{\sin\alpha}\right)^\nu \phi_{j-1-\nu}(l_2) \\
\phi_j(l_2) &= \frac{\phi_0(l_2)}{\phi_0(l_1)} \sum_{\nu=0}^{j} \frac{(-1)^\nu}{\nu!} \left(\frac{\alpha}{\sin\alpha}\right)^\nu \hat{F}_{j-\nu}(l_1)
\end{aligned}
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\end{aligned}
$$

Will one of these values be negative for some j?

For which α?

Classical trick for recursions: Coefficients of power series!

$$F(X) := \sum_{j=0}^{\infty} F_j X^j \quad \text{and} \quad \phi(X) := \sum_{j=0}^{\infty} \phi_j X^j$$

$$\text{where } F_j := F_j(l_1) \ \text{ and } \ \phi_j := \phi_j(l_2)$$

6. and 7. Formal power series equation

$$\sum_{j=0}^{\infty} F_j X^j = F(X) = \frac{e^{qX} - rX}{e^{wX} - sX}$$

where q, r, w, s functions of α and $\alpha = \arccos\left(\frac{1}{v}\right)$

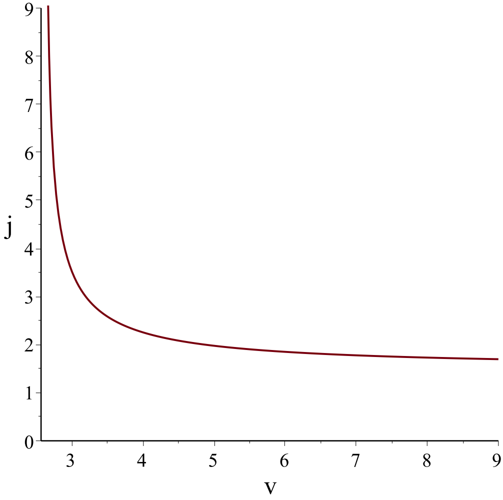
6. and 7. Formal power series equation

$$\sum_{j=0}^{\infty} F_j X^j = F(X) = \frac{e^{qX} - rX}{e^{wX} - sX}$$

where q, r, w, s functions of α and $\alpha = \arccos\left(\frac{1}{v}\right)$

- Sign of coefficients F_j? Classical approach fails!
- Ph. Flajolet: Analytic combinatorics
- Singularities of the right hand side *function*!
- Denominator $e^{wX} - sX$:
  No real singularities for $\alpha > \alpha_c \approx 1.1783$ ($v \approx 2.6144$)
- Pringsheim's Theorem: Some F_j is negative for $\alpha > \alpha_c$!
  Success and quantified version by Cauchy's Residue Theorem!

Number of rounds as function of speed



Results and open problems

Theorem

- *To enclose the fire a spiralling sprategy requires speed* $v_c > \frac{1+\sqrt{5}}{2} \approx 1.618$
- *The FollowFire strategy is successful if* $v > v_c \approx 2.6144$
- *As* v *decreases to* v_c*, the number of rounds tends to* ∞

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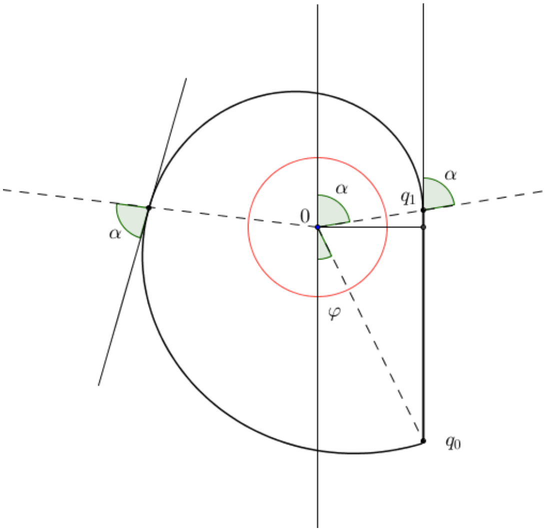
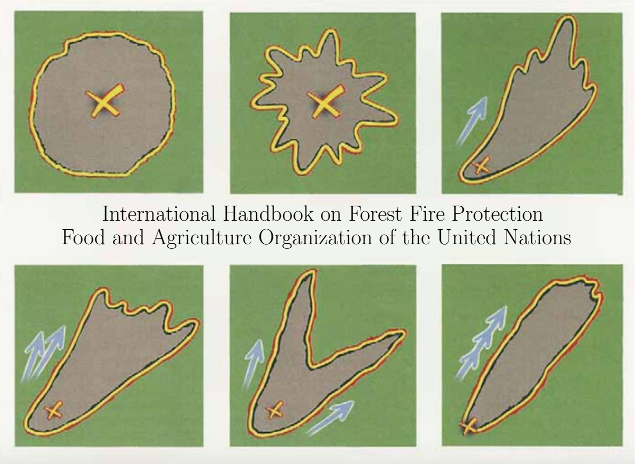
- Starting points away from the fire?

Single round for $v > 3.7788$



Different fire shapes



More open problems

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- using existing fire breaks

More open problems

- using existing fire breaks
- containing & extinguishing fire

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- using existing fire breaks
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- cooperating fire fighters

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- $\longrightarrow$ path planning in dynamic environments