On the Optimality of Spiral Search

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Abstract

Searching for a point in the plane is a well-known search game problem introduced in the early eigthies. The best known search strategy is given by a spiral and achieves a *competitive ratio* of 17.289… It was shown by Gal [7] that this strategy is the best strategy among all *monotone* and *periodic* strategies. Since then it was unknown whether the given strategy is optimal in general. This paper settles this old open problem and shows that spiral search is indeed optimal.

Keywords: Search games, motion planning, spiral search, competitive analysis, lower bound

1 Introduction

Search games (i. e., games where two players, a searcher and a hider, compete with each other) are studied in many variations in the last 60 years since the first work by Koopman in 1946. For example, Bellman [3] introduced the search for an immobile hider located on the real line with a known probability distribution, Gal [7] and independently Baeza-Yates et al. [2] solve this problem for a uniformly distributed location of the hider. The book by Gal [7] and the reissue by Alpern and Gal [1] gives a comprehensive overview on results on search games.

For analysing the efficiency of a search startegy we use the competitive framework which was introduced by Sleator and Tarjan [10], and used in many settings since then, see for example the survey by Fiat and Woeginger [6] or, for the field of online robot motion planning, see the surveys [9, 8].

We consider a special search game problem introduced by Gal [7], namely *searching for a point in the plane*. Starting from a fixed origin O we move along a path Π through the plane. Let us assume that there is an unknown target point t and let p_t denote the first point on Π so that t lies on the line segment between O and p_t. We detect t at point p_t. This means, that we *sweep* the plane until finally the unknown target t is found, see Figure 1. We assume that the target point is at least one step away from the start.

The efficiency of the search path Π is given by the worst-case target, $C := \sup_t \frac{|\Pi_O^{p_t}|}{|Ot|}$, the constant C is called the *competitive ratio* of Π. It was shown by Gal [7] that a spiral strategy is the best strategy among all *monotone* and *periodic* strategies. A strategy S represented by its radius vector $X(\theta)$ is called periodic and monotone, if θ is always increasing and X also satisfies $X(\theta+2\pi) \geq X(\theta)$. Gal states that it might be a *complicated task* to show that there is a periodic and monotone optimal strategy, a lower bound remains open.

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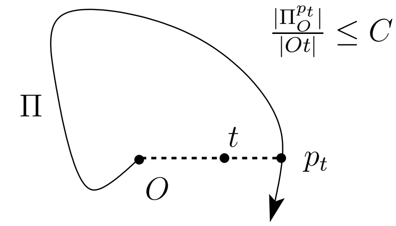

Figure 1: Searching for a point in the plane.

2 Spiral search

We consider a logarithmic spiral, Π, which is given in polar coordinates by $(\varphi, \cdot e^{\varphi \cot(\alpha)})$ for $-\infty < \varphi < \infty$, see Figure 2 for an example with $\alpha \leq \pi/2$. The angle $\alpha \leq \pi/2$ expresses the excentricity of the spiral. The length of the spiral from the center O to some point q is given by $\frac{1}{\cos\alpha}|Oq|$, for details see [4].

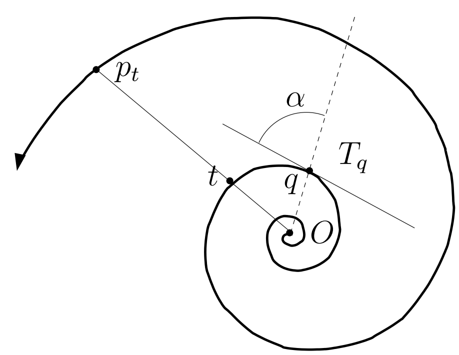

Figure 2: A logarithmic spiral and the worst-case situation.

The spiral expands successively and for every target point t there will be a first point p_t on the spiral so that the segment p_tO will hit the target t for the first time. Obviously, the worst case for the competitive ratio is given, if we miss the target t arbitrary close to

$(\varphi, e^{\varphi \cot \alpha})$ and detect t at $p_t = (\varphi + 2\pi, e^{(\varphi+2\pi)\cot\alpha})$, see Figure 2. Altogether, the worst-case ratio for the spiral is given by

$$\frac{|\Pi_O^{p_t}|}{|Ot|} = \frac{1}{\cos\alpha} e^{2\pi \cot \alpha}.$$

Optimization on α gives a competitive ratio of $17.289\ldots$ for $\cot\alpha = 0.15540\ldots$ This strategy is the best strategy among all *periodic and monotone* strategies, see Alpern and Gal [1].

Theorem 1 *The optimal spiral for searching a point in the plane achieves a competitive ratio of* $17.289\ldots$

3 Lower bound construction

The maximal lower bound known so far is $17.079\ldots$ and was presented in [5] in the context of searching for rays in the plane. The lower bound construction here consists of the following steps.

1. We consider a discrete version of the problem using a bundle of m rays that emanate from the origin, this was also used in [5].
2. We simplify the problem for m rays in some steps which results in a ratio not greater than the continous version of the problem.
3. We show that the optimal solution of the simplified problem has to be monotone and periodic for every m.
4. For every m we compute the optimal strategy using the framework of Gal. The optimal competitive ratio goes to $17.289\ldots$ if m goes to infinity.

Let us first consider a discrete version of the problem using a bundle of m rays that emanate from the origin and which are separated by an angle $\alpha = \frac{2\pi}{m}$, see Figure 3. The target will be on one of the rays. Again, the goal is detected, if it is swept by the radius vector of the trajectory, i.e., t is hit by a segment p_tO and p_t is visited on the corresponding ray. Note that if m goes to infinity we are back to the continous version of the problem. But we can neither assume that we have to visit the rays in a periodic order nor that the depth of the visits increases in every step.

We represent a search strategy, S, as follows: In the ith step, the searcher hits a ray—say ray l—at distance x_i from the origin, moves a distance $\beta_i x_i - x_i$ along the ray l, and leaves the ray at distance $\beta_i x_i$ with $\beta_i \geq 1$. Then, it moves to the next ray within distance $\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1} \cos\gamma_{i,i+1} + x_{i+1}^2}$, see Figure 3. Note that any search strategy for our problem can be described in this way. Let us assume that the ray l was visited up to distance $\beta_k x_k$ and is visited the next time at index J_k. The worst case occurs if the searcher slightly misses the goal while visiting ray l up to distance $\beta_k x_k$. Instead, it finds the goal at

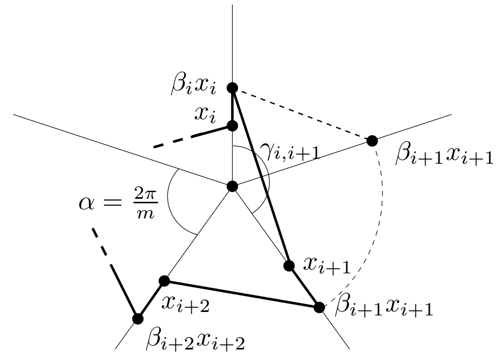

Figure 3: A bundle of rays, a reasonable strategy and a shortcut.

step x_{J_k} on ray l arbitrarily close to $\beta_k x_k$. Either we have $x_{J_k} > \beta_k x_k$; that is, the searcher discovers the goal in distance x_{J_k} on ray l, or we have $x_{J_k} < \beta_k x_k$. In the latter case, the searcher moves $\beta_k x_k - x_{J_k}$ from x_{J_k} and finds the goal by accident. Altogether, the competitive ratio, $C(S)$, is greater than

$$\frac{\sum_{i=1}^{J_k-1} \beta_i x_i - x_i + \sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2}}{\beta_k x_k}. \quad (1)$$

We simplify the problem for m rays in some steps. We do not change the movement of the strategy but we will improve the ratio. Instead of the distance

$$\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i x_{i+1}\cos\gamma_{i,i+1} + x_{i+1}^2} + \beta_{i+1}x_{i+1} - x_{i+1}$$

from $\beta_i x_i$ to x_{i+1} and then to $\beta_{i+1}x_{i+1}$ between two arbitrary successive rays we let this distance *shrink* to $\sqrt{(\beta_i x_i)^2 - 2\beta_i x_i \beta_{i+1} x_{i+1} \cos\frac{2\pi}{m} + (\beta_{i+1}x_{i+1})^2}$. This would be the distance between two neighboring rays without slipping along the second ray, see the dashed line in Figure 3. The new distance is obviously not greater than the original one, of course it might be the same for $\beta_{i+1} = 1$ and two neighboring rays, which means $\gamma_{i,i+1} = \frac{2\pi}{m}$. We change only the path length for the ratio but we do not change the movements of the given strategy. Therefore, the ratio (1) cannot increase, because the numerator will not increase.

There is only one problem in this reformulation concerning the last value of the sum in the numerator of ratio (1). The last step of the strategy (before detecting the goal at $\beta_k x_k$) goes from $\beta_{J_k-1}x_{J_k-1}$ to x_{J_k} and not directly to $\beta_{J_k}x_{J_k}$ and this step might indeed be smaller than the distance from $\beta_{J_k-1}x_{J_k-1}$ to $\beta_{J_k}x_{J_k}$. Therefore, we allow—only for the computation of the ratio—that the last value in the numerator of (1) is given by the distance of the shortest path from $\beta_{J_k-1}x_{J_k-1}$ to a neighboring ray, which is $\beta_{J_k-1}x_{J_k-1}\sin\frac{2\pi}{m}$. One can imagine that this last step will have no influence if we let m go to infinity at the end. Again, we do not change the movement of the strategy, we only improve the ratio.

For convenience, from now on we denote $\beta_i x_i$ by y_i. Altogether, we would like to minimize

$$\frac{y_{J_k-1} \sin \frac{2\pi}{m} + \sum_{i=1}^{J_k-2} \sqrt{y_i^2 - 2y_i y_{i+1} \cos \frac{2\pi}{m} + y_{i+1}^2}}{y_k}. \tag{2}$$

Obviously, the ratio (2) is never greater than ratio (1), only the numerator might be smaller.

The simplification above results in the following much more simple interpretation of the problem. We can assume that there are only two rays with angle $\frac{2\pi}{m}$ between them and the agent moves successively from one to the other, see an example (bold line) in Figure 4 for $m = 5$ up to 11 steps. For every visit the agent only decides which ray, l, of the original m rays it would visit in the m-ray setting. Note that this also means that the important indices J_k are fixed in that way. Remember that the ray visited with depth x_k is still visited the next time at index J_k.

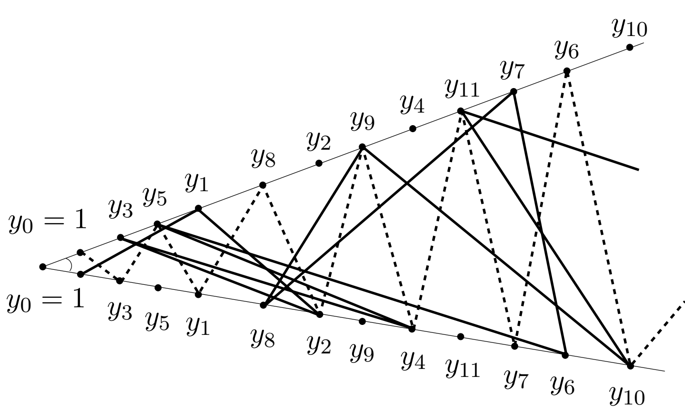

Figure 4: The problem for $m = 5$ rays interpreted by successive visits on two rays. The Strategy S (bold line) can be replaced by the sequence S' (dashed line).

Let us now assume that we have an optimal strategy $S = (y_1, y_2, y_3, y_4, \ldots)$ and given values J_k for this problem. The worst-case ratio of S is the worst-case ratio of (2). The target is at least one step away from the origin. The starting depth on every ray is 1. At every step y_i the rays are visited up to a certain depth. For the optimal sequence S we can now choose a *visiting order* (i.e., we improve the indices J_k) in such a way that at every step y_i the ray with the smallest current depth is visited next. Obviously, this will keep the ratio (2) as small as possible regardless how the original visiting order was. If a ray with smallest current depth is visited later, some more steps were made and the ratio (2) will increase. Note that we do not change the movements of the strategy.

For example, in Figure 4 in the optimal visiting order the first five steps $y_1, \ldots, y_5$ visits the five *imaginary* rays successively since the starting depth on every ray was 1. For distance y_6 we set $J_3 = 6$ since y_3 is the smallest current depth on all rays. This means that y_6 visits the same ray as y_3. Further on we obtain $J_5 = 7$ since y_5 is the smallest current depth at step y_7, then we have $J_1 = 8$, $J_8 = 9$, $J_2 = 10$, $J_9 = 11$ and so on. This is the best visiting order for S.

If two rays have exactly the same current depth, we choose the one which was visited earlier. Note that we do not change the movements of the strategy and it might still happen that y_{i+1} is smaller than y_i. But we know that at step y_{i+1} there is at least one ray that has current depth smaller than y_{i+1}, otherwise with distance y_{i+1} we would not detect more goals and step y_{i+1} was needless, we can skip such a step. Of course, after a step with $y_{i+1} < y_i$ the value y_{i+1} might be the smallest current depth on all rays. Then its ray should be visited again in the next step y_{i+2}. We simply allow to visit the same ray in two successive steps which is not possible in the original m-ray version. This additional freedom can only improve the ratio (2).

Since J_k is the index where the ray of y_k is visited next, for $J_l < J_j$ we now have $y_l \leq y_j$. The next idea is that we really visit the two rays in an *increasing* order. In the example of Figure 4 the strategy $S = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11} \ldots)$ gives $J_3 = 6$, $J_5 = 7$, $J_1 = 8$, $J_8 = 9$, $J_2 = 10$, $J_9 = 11$ and so on. Note that the first $5 = m$ elements of S visit the 5 rays successively. Thus, the sequence S now will be reordered to $S' = (y'_1, y'_2, y'_3, y'_4 \ldots)$ with $y'_1 = y_3$, $y'_2 = y_5$, $y'_3 = y_1$, $y'_4 = y_8$, $y'_5 = y_2$ $y'_6 = y_9$, $y'_7 = y_4$, $y'_8 = y_{11}$, $y'_9 = y_7$ and so on.

For S' we can again use a *visiting order* induced by the smallest current depth, this can not increase the ratio for S' as mentioned before. This means that the rays are now visited in successive order and with increasing distance. For index n in S', the index J'_n is exactly $n + m$. Thus, S' is monotone and periodic and the ratio is given by

$$\frac{y'_{n+m-1} \sin \frac{2\pi}{m} + \sum_{i=1}^{n+m-2} \sqrt{{y'_i}^2 - 2y'_i y_{i+1} \cos \frac{2\pi}{m} + {y'_{i+1}}^2}}{y'_n}. \tag{3}$$

We still have to show that S' is not worse than S with respect to the competitive ratio. The worst case for depth y_k on S is attained for the next visit at index J_k. In S' the distance y_k might be visited in an earlier step than in S, that is $y'_n = y_k$ and $n \leq k$. For index n in S', the index J'_n equals exactly $n + m$ as already seen. We would like to compare the ratios for y_k in S and $y'_n = y_k$ in S'. It is easy to see that for the sequence S also $J_k = n + m$ has to be fulfilled, since the visit order in S was induced by current smallest depth. For example, in the sequence S' in Figure 4, the ray of $y_2' = y_5$ was visited at index 7 again and in the sequence S the ray of y_5 was also visited again at index 7. Thus, for comparing the ratios for $y_2' = y_5$ we have to use the sums in the ratios of S and S' up to the same index 7 but with some different elements.

This holds in general. We consider the sequence S. Since in the beginning the m rays were visited successively by the depths $y_1, \ldots, y_m$ one of the rays is visited with depth y_{m+1} in the next step and y_{m+1} is one of the new current depths of the rays. Generally at every step y_k, the element y_k will be used on one of the rays and therefore one current depth is exchanged by y_k. In the sorted order of $y_1, \ldots, y_{k-1}$ the $k-m$ greatest element will be exchanged, which is element ${y'}_{k-m}$. Moreover, up to index k the sequences S and S' will have at most $m-1$ different entries and these entries are greater in S than in S'.

Altogether, this means that the number of visits for the worst case at y_k or $y_n' = y_k$ is the same (namely $J_k = n + m = J_n'$) and only at most $m-1$ elements in S are greater than that of S' up to index $n+m$.

We consider a simple shortest path problem. Let a sequence of elements $S = (y_0, y_1, y_2, y_3, \ldots, y_{n+m})$ be given. We use images of the corresponding points on both rays as already depicted in Figure 4. The task is, to compute a shortest path that starts at the smallest element y_j in S and visits exactly one of the two images for every element y_i but changes sucessively from one ray to the other.

We would like to show that the shortest path has to visit the rays in an increasing order, see the dashed path in Figure 4. This can be shown by induction on the length of S. For two elements in S this is trivial. So we consider a sequence S with $n+1$ elements and assume that the statement holds for all sequences with less than n elements. By triangle inequality we can show that the shortest path always starts with the segment of smallest slope. Now we delete the smallest element y_j out of S and the corresponding shortest path visits the remaining elements in increasing order starting at the second smallest value y_i. The image of y_j is closer to y_i than to any other part of the shortest path for $S \setminus \{y_j\}$, therefore we can combine the segment $y_j y_i$ with the shortest path starting at y_i. We obtain an overall shortest path that visits the images in increasing order.

Unfortunately, S and S' might have $m-1$ different elements. All these elements in S are greater than the corresponding elements in S'. The corresponding shortest path problem for S will result in a path of smaller length, if we move a couple of points with greatest distance closer to the origin. Therefore, we substitute the differing elements of S by the corresponding elements of S'.

Altogether, we conclude that $y_{n+m-1} \sin \frac{2\pi}{m} + \sum_{i=1}^{n+m-2} \sqrt{y_i^2 - 2y_i y_{i+1} \cos \frac{2\pi}{m} + y_{i+1}^2}$ is greater than or equal to

$$y_{n+m-1}' \sin \frac{2\pi}{m} + \sum_{i=1}^{n+m-2} \sqrt{{y_i'}^2 - 2y_i' y_{i+1} \cos \frac{2\pi}{m} + {y_{i+1}'}^2}$$

and the sequence S' is optimal because for every $y_n' = y_k$ the ratio (3) is not greater than the ratio (2).

In principle, we are already done, because as m goes to infinity we get arbitrarily close to the *searching-for-a-point-in-the-plane* problem and there is always an optimal solution that is periodic and monotone. Thus, together with the previous result of Gal, spiral search is optimal.

But we can also proceed more directly as follows: We compute an optimal sequence S' for ratio (3). Fortunately, S' is periodic and monotone and fulfills some other nice properties (for example unimodality) so that a general framework of Gal is applicable, see [7, 1]. This means that (3) is minimized by an exponential sequence $y_i' = a^i$. Simple arithmetic shows that the ratio is given by $f(a, m) = a^{m-1} \sin \frac{2\pi}{m} + \frac{a^{m-1}}{a-1} \sqrt{1 - 2a \cos \frac{2\pi}{m} + a^2}$. Thus, we can analytically find the value $a_{\min}$ that minimizes $f(a, m)$. For increasing m the corresponding value $f(a_{\min}, m)$ converges to 17.289... For example, for $m = 5000$ we compute $a_{\min} = 1.000195303\ldots$ and $f(a_{\min}, 5000) = 17.289\ldots$

Theorem 2 *Spiral search is optimal.*

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