Searching for an axis-parallel shoreline

Elmar Langetepe

University of Bonn, Computer Science I, Römerstraße 164, 53117 Bonn, Germany.

Abstract. We are searching for an unknown horizontal or vertical line in the plane under the competitive framework. We design a well-suited framework for all *cylcic* strategies that result in two-sequence functionals. For optimizing such functionals we apply a method that combines two main paradigms. The given solution shows that the combination method is of general interest. Finally, we obtain the current best strategy and can prove that this is the best strategy among all cyclic strategies which is a main step toward a lower bound construction.

Keywords: Search games, computational geometry, motion planning, online algorithm, competitive analysis, optimizing two-sequence functionals

1 Introduction

Let us assume that we are lost at sea without sight and we are searching for an unknown shoreline in the competitive sense. That is, we compare the lenght of our search path until arriving the shoreline to the length of the shortest path to the shoreline if it was known in advance. The logarithmic spiral conjecture says that the best strategy is a logarithmic spiral and the best spiral achieves a competitive ratio of $13.81113\ldots$, see [2, 3, 5, 6]. This old fundamental search problem of searching for a line in the plane has recently attracted new attention.

On the one hand it was recently shown that spiral search is optimal for the *searching-for-point-in-the-plane* scenario [16]. This result gives hope that it will be possible to prove that the logarithmic spiral conjecture is also true if we are searching for a line.

On the other hand there is a new upper bound on the competitive ratio for finding an axis-parallel shoreline [11]. The given strategy makes use of a special representation that has to be optimized. Finally, the strategy achieves a ratio of $12.5406\ldots$

In this paper we make use of an elegant representation of all cyclic strategies in the axis-parallel shoreline setting. The problem results in the optimization of two-sequence functionals. For optimizing two-sequence functionals we can apply a combination of two main paradigms. The given example shows that this generic approach is of general interest, see also [12, 17].

We easily achieve the same ratio as in [11] by considering a special case of our representation. Furthermore, we can slightly improve the above result and we can also state that for *cyclic* strategies there is no hope for further improvements. The main open question is whether the given strategy is optimal in general. But now we only have to show that there is always an optimal cyclic strategy.

2 Preliminaries

We are searching for an unknown line l that is parallel to one of the axes. Since l can be everywhere in the plane any search path Π finally visits all possible lines l. Consider the length of a search path Π from the origin o to the first point p_l where some line l is met. Let $\Pi_o^{p_l}$ denote this path and $|\Pi_o^{p_l}|$ its distance. Competitive analysis compares $|\Pi_o^{p_l}|$ to the length of the shortest path from o to l, denoted by $|ol_\perp|$. The worst-case location of l gives the competitive ratio of the strategy which means $C = \sup_l \frac{|\Pi_o^{p_l}|}{|ol_\perp|}$.

Competitive analysis was introduced by Sleator and Tarjan [22], and used in many settings since then, see for example the survey by Fiat and Woeginger [4] or, for the field of online robot motion planning, see the surveys [10, 19]. Note that we assume that the unknown line is at least one step away from the origin. Otherwise we have to introduce a fixed additive constant. Both interpretations are equivalent, see [22].

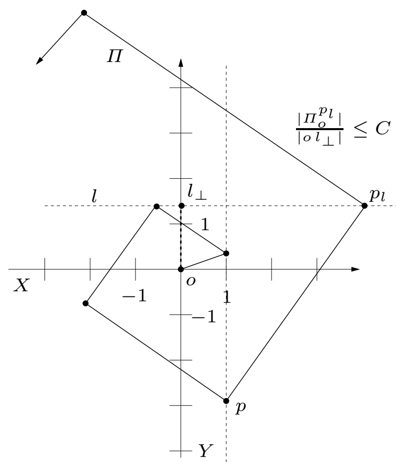

Fig. 1. The strategy of Jeż and Łopuszański [11]. The current worst case occurs at kinks of the strategy, see the hit points p_l and p.

It seems that any reasonable strategy should visit the four possible directions in a cyclic order. In [11] such a strategy is defined by a sequence of points (x_i, y_i) for $i = 1, 2, 3, \ldots$ These points are visited in the given order. More precisely Jeż and Łopuszański define a strategy by $x_{2k+1} = -\alpha\, x_{2k-1}$, $x_{2k} = x_{2k-3}$, $y_{2k+2} = -\alpha\, y_{2k}$ and $y_{2k+1} = y_{2k-2}$ with an expanding factor α. The reason for this special formulation is that they would like to let the projections onto the axes expand by a factor α. In Fig. 1 there is an example of the strategy for the best α with starting values $x_1 = 1$ and $x_2 = \sqrt{a}$. For $\alpha = 2.03\ldots$ the strategy achieves a ratio of $12.5406\ldots$

While the strategy is running there is a current depth up to which all lines in a given direction already have been visited. The smallest current depth among all directions will be responsible for a worst case ratio in the very near future. Let us call this the *current* worst case situation. The current worst case occurs at a discrete moment when the corresponding line is met again. Let us call this point a worst case *hit point*.

The current worst case situation for the strategy in Fig. 1 is always attained exactly at a kink at coordinate $x_{2k} = x_{2k-3}$ or $y_{2k+1} = y_{2k-2}$. In this sense the strategy behaves in a very *symmetric* way. It visits the current smallest Y-coordinate (respectively X-coordinate) and also ends its expansion in the X-coordinate (respectively Y-coordinate) here, thus making a kink, see the hit points p_l and p in Fig. 1.

3 A unique design of cyclic strategies

We would like to make use of a more intuitive and more general representation of a cyclic strategy. First, we consider symmetric strategies and three cases, see Fig. 2. Then we show that all three cases and also its combination can be described uniquely by an appropriate interpretation of the first case.

In the first case the kink of the strategy always happens before the current hit point is visited. In the second case the kink of the strategy happens after the current hit point was visited. For completeness, in the third case we consider the same situation as in Fig. 1, kink and current hit point coincidence.

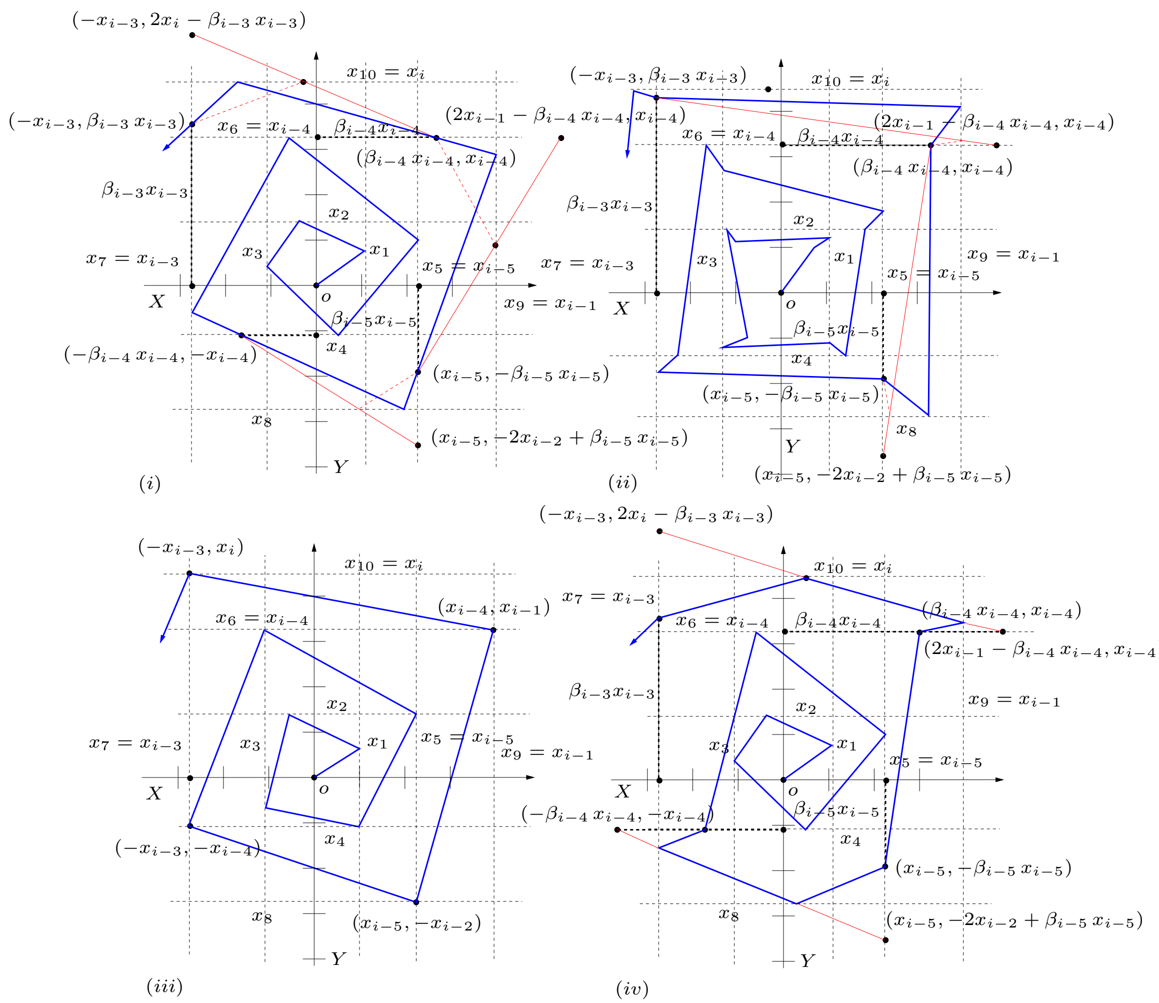

Fig. 2. Representation of symmetric strategies and calculation of the ratio. (i) The strategy always makes a kink before the current worst case occurs at a hit point. (ii) The strategy always makes a kink after the current worst case occurs. (iii) Kink and current worst case coincidence. (iv) The elegant description of all combinations of the former three cases.

Our representation of a strategy is as follows. We consider a fixed cyclic visiting order east, north, west, south, east and so on and an infinite sequence of positive values $(x_1, x_2, x_3, \ldots)$ that represent the distances to the corresponding lines. For simplicity, we also denote the corresponding lines by these distances.

In the first case, where the kink happens at line x_i before the hit point at line x_{i-3} is visited, we have to parameterize the Y-coordinate (respectively X-coordinate) of the hit point. Therefore we consider a positive value $\beta_{i-3}x_{i-3}$ for this distance and for all all other points in an analogous way, see also Fig. 2(i). This means that we have a second positive sequence $(\beta_1, \beta_2, \beta_3, \ldots)$.

For calculating the ratio for all current worst case situations in this scenario we simply have to sum up the distance between two hit points on the lines x_{i-4} and x_{i-3}. In between the line x_i has to be visited. Let us assume that we have a situation as depicted in Fig. 2(i). We can optimize such a strategy if we simply consider the shortest path that visits the line x_i between two fixed hit points. Therefore we calculate this distance by mirrowing the hit point of x_{i-3} at line x_i and consider the distance between the image and the hit point at x_{i-2}. In the situation of Fig. 2(i), we have to calculate the distance between the points $(-x_{i-3}, 2x_i - \beta_{i-3}\, x_{i-3})$ and $(\beta_{i-4}\, x_{i-4}, x_{i-4})$ which gives

$$\sqrt{(x_{i-3} + \beta_{i-4}\, x_{i-4})^2 + (2x_i - \beta_{i-3}\, x_{i-3} - x_{i-4})^2}\,.$$

Fortunately, this distance can be described in the same way for all i. For example, consider the movement between the hit points on x_{i-4} and x_{i-5} that visits x_{i-1} in between. The minimal length of this path is the length of the line between $(2x_{i-1} - \beta_{i-4}\, x_{i-4}, x_{i-4})$ and $(x_{i-5}, -\beta_{i-5}\, x_{i-5})$ which gives

$$\sqrt{(x_{i-4} + \beta_{i-5}\, x_{i-5})^2 + (2x_{i-1} - \beta_{i-4}\, x_{i-4} - x_{i-5})^2}\,.$$

Altogether we have a unique formula of the optimal distances between two hit points. The strategy should make use of such a specular reflection between two hit points. Therefore our task is to find two infinite positive sequences $\beta := (\beta_1, \beta_2, \beta_3, \ldots)$ and $X := (x_1, x_2, x_3, \ldots)$ that minimizes the ratio

$$\sup_k \frac{\sum_{i=-2}^{k} \sqrt{(x_i + \beta_{i-1}\, x_{i-1})^2 + (2x_{i+3} - \beta_i\, x_i - x_{i-1})^2}}{x_k}\,. \tag{1}$$

Here for the starting round we simply set $x_0 = x_{-1} = x_{-2} = 1$ and $x_{-3} = 0$ since in the first round the current worst case occurs at lines at distance 1 (it was assumed that the line is at least one step away). There is some freedom for setting β_{-2}, β_{-1} and β_0 and we extend the sequence β accordingly. The ratio in (1) is called a functional with two infinite sequences.

With similar arguments we find a functional for the second case, where the strategy makes a kink after the hit point for the current worst case, see Fig. 2(ii). Now the movement between two hit points at x_{i-4} and x_{i-3} visits and reflects at line x_{i-1}. With the notation and the reflection idea from Fig. 2(ii) we have to calculate the length of the segment between $(-x_{i-3}, \beta_{i-3}\, x_{i-3})$ and $(2x_{i-1} - \beta_{i-4}\, x_{i-4}, x_{i-4})$. This results in the distance

$$\sqrt{(2x_{i-1} - \beta_{i-4}\, x_{i-4} + x_{i-3})^2 + (\beta_{i-3}\, x_{i-3} - x_{i-4})^2}\,.$$

Again, this formula describes all occuring distances and the task is to find two infinite positive sequences $\beta := (\beta_1, \beta_2, \beta_3, \ldots)$ and $X := (x_1, x_2, x_3, \ldots)$ that minimizes the ratio

$$\sup_k \frac{\sum_{i=-2}^{k} \sqrt{(2x_{i+2} - \beta_{i-1}\, x_{i-1} + x_i)^2 + (\beta_i\, x_i - x_{i-1})^2}}{x_k}\,. \tag{2}$$

For the starting round we make use of $x_{-1} = x_{-2} = 1$ and $x_0 = x_{-3} = 0$ and we have some freedom for the values β_{-2}, β_{-1} and β_0, thus again β is extended accordingly.

The third case, where kink and hit point coincidence, is handled analogously and results in the following ratio. Fortunately here we have to find a single sequence $X := (x_1, x_2, x_3, \ldots)$ that minimizes

$$\sup_k \frac{\sum_{i=-2}^{k} \sqrt{\left(x_{i+3}-x_{i-1}\right)^2+\left(x_{i+2}+x_i\right)^2}}{x_k}\,. \tag{3}$$

Here we make use of a starting value $x_{-3}=0$ and we have some freedom for the values x_{-2}, x_{-1} and x_0 and X is extended by these values.

In the remaining case we consider all combinations of the first three cases which means that sometimes the hit point of x_i can be visited before, after or on x_{i+3}. We show that the description of such a strategy is the same as in (1), if we allow that $\beta_i\, x_i$ goes beyond x_{i+3}. In Fig. 2(iv) there is no kink at x_9 between the hit points at x_5 and x_6. Fortunately, we are able to describe the corresponding movement in the same way as in case Fig. 2(i). More generally, we have to calculate the distance between the points $(-x_{i-3}, 2x_i-\beta_{i-3}\, x_{i-3})$ and $(\beta_{i-4}\, x_{i-4}, x_{i-4})$ which gives

$$\sqrt{\left(x_{i-3}+\beta_{i-4}\, x_{i-4}\right)^2+\left(2x_i-\beta_{i-3}\, x_{i-3}-x_{i-4}\right)^2}\,.$$

Additionally, the distance between $(2x_{i-1}-\beta_{i-4}\, x_{i-4}, x_{i-4})$ and $(x_{i-5}, -\beta_{i-5}\, x_{i-5})$ again gives

$$\sqrt{\left(x_{i-4}+\beta_{i-5}\, x_{i-5}\right)^2+\left(2x_{i-1}-\beta_{i-4}\, x_{i-4}-x_{i-5}\right)^2}\,.$$

This means that (1) subsumes all three cases, if we allow $\beta_i\, x_i > x_{i+3}$, $\beta_i\, x_i = x_{i+3}$ and $\beta_i\, x_i < x_{i+3}$. Obviously, our next task is to optimize two-sequence functionals stemming from (1).

4 Optimization of single sequence functionals

There are two main paradigms for computing optimal strategies by optimizing single sequence functionals. In this section we briefly repeat the main ideas and and exemplify the application of the first approach to the functional

$$F_k(x_{-2}, x_{-1}, x_0, x_1, \ldots) := \frac{\sum_{i=-2}^{k} \sqrt{\left(x_{i+3}-x_{i-1}\right)^2+\left(x_{i+2}+x_i\right)^2}}{x_k}\,, \tag{4}$$

that stems from ratio (3). More precisely, we are searching for an infinite strategy $X = x_{-2}, x_{-1}, x_0, x_1 \ldots$ so that

$$\inf_Y \sup_k F_k(Y) = C \text{ and } \sup_k F_k(X) = C\,.$$

4.1 Optimality of the exponential function

The following theorem states that the supremum of a functional is minimized by an exponential function, if certain properties are fulfilled. The given problem results in an optimization problem for functionals $F_k(X)$ with infinite sequences $X = (x_1, x_2, x_3, \ldots)$. For two sequences $X = (x_1, x_2, x_3, \ldots)$ and $Y = (y_1, y_2, y_3, \ldots)$ let $X+Y := (x_1+y_1, x_2+y_2, x_3+y_3, \ldots)$ and $A\cdot X := (A\cdot x_1, A\cdot x_2, A\cdot x_3, \ldots)$ for a constant A.

Theorem 1. *(adapted from Gal [7, 6], Alpern and Gal [1] and Schuierer [21])*
Given a sequence of functional $F_k(X)$ for all $k \geq k_0$ and infinite sequences $X = (x_1, x_2, x_3, \ldots)$ and $Y = (y_1, y_2, y_3, \ldots)$ with $x_i > 0$ and $y_i > 0$.
If the following conditions hold for F_k:

(i) F_k is continous,

(ii) F_k *is* unimodal, *which means:* $F_k(A \cdot X) = F_k(X)$ *and* $F_k(X + Y) \le \max\{F_k(X), F_k(Y)\}$,

(iii)

$$\liminf_{a\to\infty} F_k\left(\frac{1}{a^k}, \frac{1}{a^{k-1}}, \dots, \frac{1}{a}, 1\right) = \liminf_{\varepsilon_k,\varepsilon_{k-1},\dots,\varepsilon_1\to 0} F_k\left(\varepsilon_k, \varepsilon_{k-1}, \dots, \varepsilon_1, 1\right),$$

(iv)

$$\liminf_{a\to 0} F_k\left(1, a, a^2, \dots, a^k\right) = \liminf_{\varepsilon_k,\varepsilon_{k-1},\dots,\varepsilon_1\to 0} F_k\left(1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \right),$$

(v) $F_{k+1}(x_1, \dots, x_{k+2}) \ge F_k(x_2, \dots, x_{k+2})$.

then

$$\sup_k F_k(X) \ge \inf_a \sup_k F_k(A_a)$$

with $A_a = a^0, a^1, a^2, \dots$ *und* $a > 0$. *The supremum of the functional is minimized by an exponential function.*

Note that the given form of condition (v) is a replacement shown by Schuierer [21], see also Alpern and Gal [1]. Obviously, the functional of (4) fulfills condition (i), (iii), (iv), (v) and the part $F_k(A \cdot X) = F_k(X)$ of (ii). The remaining task is to prove that unimodality holds in the additive sense which is $F_k(X + Y) \le \max\{F_k(X), F_k(Y)\}$. Normally, this is the most difficult thing, here it is easy.

The triangle inequality of the vectors $(a - b, c + d)$ and $(e - f, g + h)$ gives

$$\sqrt{(a-b)^2 + (c+d)^2} + \sqrt{(e-f)^2 + (g+h)^2} \ge \sqrt{(a+e-(b+f))^2 + (c+d+g+h)^2}\,.$$

Now let $F_k(X) \le K$ and $F_k(Y) \le K$, which is also true for $K := \max\{F_k(X), F_k(Y)\}$. We have

$$\frac{\sum_{i=-2}^{k}\sqrt{\left(x_{i+3} - x_{i-1}\right)^2 + \left(x_{i+2} + x_i\right)^2}}{x_k} + \frac{\sum_{i=-2}^{k}\sqrt{\left(y_{i+3} - y_{i-1}\right)^2 + \left(y_{i+2} + y_i\right)^2}}{x_k} \le K \cdot (x_k + y_k)$$

and we can prove $F_k(X + Y) \le K$ by the inequality shown above. This gives unimodality in the additive sense for (4).

Obviously, the functional $F_k(x_{-2}, x_{-1}, \dots, x_k)$ fulfills the conditions of Theorem 1 and we conclude

$$\sup_k F_k(X) \ge \inf_a \sup_k F_k(A_a)$$

where $A_a = a^{-2}, a^{-1}, a^0, a^1, a^2, \dots$ and $a > 0$. We can substitute x_i by a^i and can make use of a geometric serie. Altogether, the problem is now solved by

$$\inf_a \frac{\sum_{i=-2}^{k}\sqrt{\left(a^{i+3} - a^{i-1}\right)^2 + \left(a^{i+2} + a^i\right)^2}}{a^k} = \min_a \frac{\sqrt{(a^4 - a^2 + 1)(a^2+1)^2}}{a - 1}$$

Thus we have found a simple function that has to be minimized over a. Optimizing the last function in a by analytic means gives $a = 1.425421\dots$ and exactly the ratio $12.54064\dots$ presented in [11]. This shows that there is no room for improvements if we choose a cyclic strategy that always makes a kink at the current worst case situation.

4.2 Equality approach

On the other hand some authors [9, 15, 18, 13, 20] suggest to adjust an optimal strategy $X = x_1, x_2, \ldots$ with $F_k(X) \leq C$ to an optimal strategy $X' = x'_1, x'_2, \ldots$ with $F_k(X') = C$ where C is the (probably unknown) best achievable factor. Then one will try to retrieve a recurrence for the values of X' from the equation $F_k(X') = C = F_{k+1}(X')$ and find the smallest C that fulfills this recurrence with positive values. In this section we do not apply this method to the functional (4) because this seems to be difficult. Instead we use the functional $F_k := \frac{\sum_{i=1}^{k} x_i}{x_k}$, that stems from the 2-ray search problem, see [2, 14].

It can be shown that for the 2-ray search problem such a strategy X with $F_k(X) = C = F_{k+1}(X)$ exists. How will we find the optimal strategy in this case? One will try to retrieve a recurrence for the values of X from the equation $F_k(X) = C = F_{k+1}(X)$.

For the 2-ray search problem we assume that $X = x_1, x_2, \ldots$ achieves equality in every step. We conclude $\sum_{i=1}^{k+2} x_i = C x_{k+1}$ and $\sum_{i=1}^{k+1} x_i = C x_k$. Subtracting both sides gives the recurrence $x_{k+2} = C(x_{k+1} - x_k)$ for $k = 1, 2, \ldots$ Obtaining positive solutions for recurrences can be solved by analytic means, see [8]. It can be shown that for $C < 4$ there is no positive sequence that fulfills the given recurrence $x_{k+2} = C(x_{k+1} - x_k)$. Furthermore, for $x_i := (i+1)2^i$ we have $x_{k+2} = (k+3)2^{k+1} = 4(x_{k+1} - x_k) = (3k+4)2^{k+2} - (k+1)2^{k+2}$. This means, that there is a positive sequence, $x_i := (i+1)2^i$ that attains the optimal value $C = 4$.

Altogether, we have two different approaches stemming from different paradigms. In the following we will combine both paradigms in order to solve two-sequence functionals.

5 Optimizing two-sequence functionals

Now we would like to find the best cyclic strategy and have to optimize ratio (1). Our task is to find optimal sequences $\beta = (\beta_{-2}, \beta_{-1}, \beta_0, \beta_1, \ldots)$ and $X = (x_0, x_1, x_2, x_3, \ldots)$ that minimizes the supremum of the following functional for all k.

$$F_k(\beta, X) := \frac{\sum_{i=-2}^{k} \sqrt{\left(x_i + \beta_{i-1}\, x_{i-1}\right)^2 + \left(2x_{i+3} - \beta_i\, x_i - x_{i-1}\right)^2}}{x_k} \tag{5}$$

Let us assume first that the sequence β is fixed. For the same reason as in the previous section we can apply Theorem 1 to the sequence X.

The triangle inequality of the vectors $(a + \beta\, b, 2c - \gamma\, d - e)$ and $(f + \beta\, g, 2h - \gamma\, i - j)$ gives

$$\sqrt{(a + \beta\, b)^2 + (2c - \gamma\, d - e)^2} + \sqrt{(f + \beta\, g)^2 + (2h - \gamma\, i - j)^2} \geq \sqrt{((a + f) + \beta\,(b + g))^2 + (2(c + h) - \gamma\,(d + i) - (e + j))^2}\,.$$

With similar arguments as in Sect. 4.1 it is clear that Theorem 1 is applicable to ratio (5). This means that a strategy $x_i = a^i$ will optimize $\sup_k F_k(X, \beta)$. But there is still a second sequence β that has to be optimized and a simple function for finding a and β is not given. Therefore we suggest to apply the second paradigm (equality approach) of Sect. 4.2 first. We would like to reduce the complexity of the problem.

Lemma 2. *For the functional (5) and the optimal sequences X and β with $\sup_k F_k(X, \beta) = C$ there always exists sequences X' and β' so that $F_l(X', \beta') = C$ is fulfilled for all $l \geq 0$.*

Proof. The proof works by induction on k, it is non-constructive because an optimal strategy is not known so far and we use a limit process for the inductive step.

Let us assume that an optimal cyclic strategy is given that minimizes $\sup_k F_k(X, \beta) = C$.

We would like to show by induction that for every k there is always a strategy X' and β' so that $F_l(X, \beta) = C$ is fulfilled for all $0 \leq l \leq k$ and $F_l(X, \beta) \leq C$ for $l > k$.

First, we show that we can let a single x_k shrink a bit. Let us assume that $F_k(X, \beta) < C$ holds. Then we can let x_k decrease to x'_k and adjust β_k to β'_k so that $F_k(X', \beta') = C$ and $F_l(X', \beta') \leq C$ for all $l \neq k$ holds.

The adjustment works as follows and is motivated in Fig. 3. If we move the line of x_k toward the origin then three *movements* of the strategy are concerned. First, the movement between the hit points on x_{k-1} to x_{k+1} that reflects on x_k. This part of the strategy will always be shorter, if we move x_k towards the origin. If one of the corresponding segments become horizontal or vertical, we proceed by moving the corresponding segments toward the origin, also.

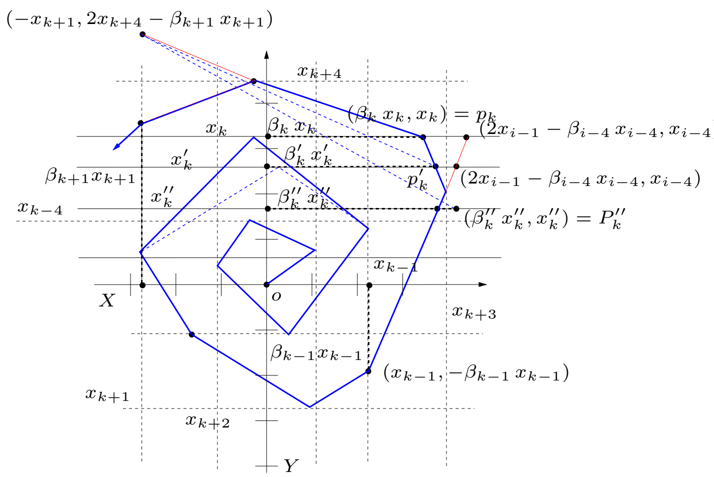

Fig. 3. If we move x_k downwards to x'_k closer to the origin, we decrease all distances which are concerned. This remains true, if we move beyond the reflection at x_{k+3}, compare x''_k. The ratio $F_k(X', \beta')$ increases, all other ratios decrease or remain the same.

Additionally, x_k and $\beta_k\, x_k$ defines a hit point p_k. The movement from p_k to the hit point on x_{k+1} that reflects on x_{k+4} and the movement of p_k toward the hit point of x_{k-1} that reflects on x_{k+3} will be concerned, if we move x_k to the origin.

Now let the line for x_k move towards the origin. Let us first assume that the kink happens on x_{k+3} before p_k is met. The other case will be handled later.

Let the intersection point p'_k with the original strategy define the new hit point. In turn it defines the value of β'. Thus, the length of the path will not increase. In fact by triangle inequality the path lenght decrease a bit because the kink appears a bit *earlier*, see the dashed path in Fig. 3.

Until reaching the reflection point at x_{k+3} using the adapted values x'_k and β'_k will always decrease the overall path lenght as indicated above. Fortunately, we can even move further on without changing the description of the ratio or functional, respectively.

We simply let the hit point move beyond x_{k+3} which means that $\beta''_k\, x''_k > x_{k+3}$ holds, see Fig. 3. The coordinates of the hit point still depends on x''_k and $\beta''_k\, x''_k$ in the same way. The movement from the hit point at x_{k-1} to the hit point at x''_k and the movement from the hit point at x''_k to the hit point at x_{k+1} is described by the same formulas as before. More precisely, by

$$\sqrt{(x_{k+1}+\beta_k''\,x_k'')^2+(2x_{k+4}-\beta_{k+1}\,x_{k+1}-x_k'')^2} \text{ and } \sqrt{(x_k''+\beta_{k-1}\,x_{k-1})^2+(2x_{k+3}-\beta_k''\,x_k''-x_{k-1})^2}\,.$$

Although the reflection on x_{k+3} now appears after p_k'' we have the same description of the strategy. That is we can move x_k' closer to the origin to x_k''. The length of the overall path further decrease. We simply adjust β_k'' accordingly.

While decreasing x_k the ratio $F_k(X, \beta)$ finally has to increase since there are always elements of the path that will not be concerned. In the induction proof we will see that finally there is some $x_k' > 0$ that let the ratio increase to $F_k(X', \beta') = C$. We will also see that we never have to move x_k below x_{k-4} in order to obtain this equality.

Therefore finally we will have $F_k(X', \beta') = C$ and $F_l(X', \beta') \leq C$ for all $l \neq k$.

Now the full proof works by induction. First, we let x_0 shrink so that $F_0(X', \beta') = C$ and $F_l(X', \beta') \leq C$ for all $l \geq 1$ holds. This is always possible because some parts of the path to the hit point on x_0 remains the same.

The induction hypothesis says that for index k there is always an adjustment of a strategy so that X' and β' exists with $F_l(X', \beta') = C$ holds for all $0 \leq l \leq k$ and $F_l(X', \beta') \leq C$ for $l > k$. Additionally, the adjustment let the values of X' shrink but they will never get zero.

Now we adjust x_{k+1} and let it shrink to x_{k+1}'. By induction hypothesis $F_{k-3}(X', \beta') = C$ holds before x_{k+1} and β_{k+1} is adjusted. This means that we will attain equality for $F_{k+1}(X', \beta') = C$ for x_{k+1}' and β_{k+1}' before x_{k+1}' reaches x_{k-3}'.

Now we have $F_{k+1}(X', \beta') = C$ and $F_l(X', \beta') \leq C$ for $l \neq k+1$. We apply the induction hypothesis again for the first k values of the new X'. We can repeat this process. This means that we will have shrinking values $(x_0', x_1', x_2', \ldots, x_{k+1}')$ but they will never get zero. Finally, they have to run into a limit that gives $F_l(X', \beta') = C$ for $0 \leq l \leq k+1$.

Thus, the inductive step is true. For all k there is an optimal strategy so that $F_l(X', \beta') = C$ for $0 \leq l \leq k$. □

Altogether, we can now apply the idea of Sect. 4.2. We will make use of the fact that there is an optimal cyclic strategy with $F_k(X, \beta) = C = F_{k-1}(X, \beta)$. This means

$$C\,(x_k - x_{k-1}) = \sqrt{(x_k+\beta_{k-1}\,x_{k-1})^2+(2x_{k+3}-\beta_k\,x_k-x_{k-1})^2}$$

and we obtain a new functional

$$G_k(\beta, X) := \frac{\sqrt{(x_k+\beta_{k-1}\,x_{k-1})^2+(2x_{k+3}-\beta_k\,x_k-x_{k-1})^2}}{x_k - x_{k-1}} \tag{6}$$

and the new task is to find sequences X and β so that $\sup_k G_k(X, \beta)$ is minimal.

Fortunately, we can apply Theorem 1 to $G_k(X, \beta)$. In the same way as indicated in the beginning of this section we can prove additive unimodality by applying triangle inequality. The remaining conditions also hold. This means that there is an optimal strategy for (6) with $x_i := a^i$ for $a > 1$.

Substituting x_i by a^i and using some simple transformation shows that

$$G_k(A, \beta) = \frac{\sqrt{(a+\beta_{k-1})^2+(2a^2-\beta_k\,a-1)^2}}{a-1}\,. \tag{7}$$

Let us assume that we have found the best a. Since β_{k-1} takes over the role of β_k if we consider G_{k+1}, the best we can do is let β_i be a constant for all i. This means that we have to optimize a function $f(a, b) := \frac{\sqrt{(a+b)^2+(2a^4-b\,a-1)^2}}{a-1}$. We would like to find the minimum of $f(a, b)$ by analytic means. The derivative of $f(a,b)$ in b gives $\frac{b+2a-2a^5+ba^2}{\sqrt{(a+b)^2+(2a^4-b\,a-1)^2}(a-1)}$. It is zero if and only if $b := 2(a^2-1)a$ holds. So for all a we have to use this b for minimization. Finally, we only have to optimize the function $f(a) := \frac{\sqrt{(a^2+1)(2a^2-1)^2}}{a-1}$. For $a > 1$ this function has a unique minimum of $12.53853842\ldots$ for $a = 1.431489\ldots$ and we have $b = 2(a^2-1)a = 3.0037344\ldots$

Note that this is only a very small improvement on the strategy of Jeż and Łopuszański but in comparison to the former result we can state that this is the best strategy that visits the directions in a cyclic order.

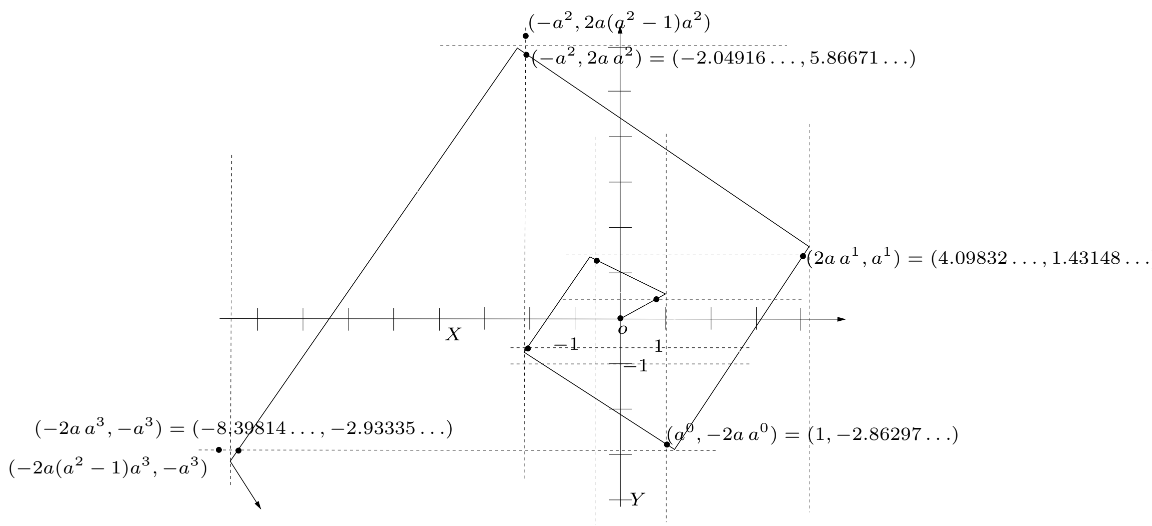

Fig. 4. The optimal cyclic strategy defined by $x_i = a^i$ and $\beta_i\, x_i = b\, a^i$ with $a = 1.431489\ldots$ and $b = 2a = 2.826979\ldots$.

Theorem 3. *An optimal cyclic strategy that finds a horizontal or vertical shoreline can be described by* $x_i = a^i$ *and* $\beta_i\, x_i = b\, a^i$ *and obtains an optimal competitive ratio of* $12.53853842\ldots$ *for* $a = 1.431489\ldots$ *and* $b = 2(a^2-1)a = 3.0037344\ldots$

The strategy is shown in Fig. 4. Note that we have $\beta_i\, x_i > x_{i+3}$ and the strategy makes a kink after the current hit point. Therefore we can also describe the optimal strategy in the fashion of (2) by computing a corresponding $b = b_i$, so that $\beta_i\, x_i < x_{i+3}$ hold. This b can be computed uniquely as follows. We compute $2(a^2-1)a - a^3 = a^3 - 2a$ and this is subtracted from a^3 and gives $b = 2a$. This means that we can describe the strategy by $b = 2a = 2.826979\ldots$ and $a = 1.431489\ldots$ also. Inserting into (2) gives the same ratio $12.53853842\ldots$ and this is exactly what we have expected. Cyclic startegies can be completely defined by (1).

6 Conclusion

Within this paper we have shown how to combine two standard methods for optimizing functionals in competitive analysis in order to optimize two-sequence functionals. More precisely, we found the current best strategy for searching a horizontal or vertical shoreline. The given approach shows that the strategy is the best strategy among all cyclic strategies. The main open question is finding a general (tight) lower bound. It remains to show that there is always an optimal strategy that visits the directions in a cyclic order.

References

[1] S. Alpern and S. Gal. *The Theory of Search Games and Rendezvous.* Kluwer Academic Publications, 2003.

[2] R. Baeza-Yates, J. Culberson, and G. Rawlins. Searching in the plane. *Inform. Comput.*, 106:234–252, 1993.

[3] M. Chrobak. A princess swimming in the fog looking for a monster cow. *SIGACT News*, 35(2):74–78, 2004.

[4] A. Fiat and G. Woeginger, editors. *On-line Algorithms: The State of the Art*, volume 1442 of *Lecture Notes Comput. Sci.* Springer-Verlag, 1998.

[5] S. R. Finch and L.-Y. Zhu. Searching for a shoreline, 2005.

[6] S. Gal. *Search Games*, volume 149 of *Mathematics in Science and Engineering.* Academic Press, New York, 1980.

[7] S. Gal and D. Chazan. On the optimality of the exponential functions for some minmax problems. *SIAM J. Appl. Math.*, 30:324–348, 1976.

[8] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison-Wesley, Reading, MA, second edition, 1994.

[9] C. Hipke, C. Icking, R. Klein, and E. Langetepe. How to find a point on a line within a fixed distance. *Discrete Appl. Math.*, 93:67–73, 1999.

[10] C. Icking, T. Kamphans, R. Klein, and E. Langetepe. On the competitive complexity of navigation tasks. In H. Bunke, H. I. Christensen, G. D. Hager, and R. Klein, editors, *Sensor Based Intelligent Robots*, volume 2238 of *Lecture Notes Comput. Sci.*, pages 245–258, Berlin, 2002. Springer.

[11] A. Jeż and J. Łopuszański. On the two-dimensional cow search problem. *Inf. Process. Lett.*, 109(11):543–547, 2009.

[12] T. Kamphans and E. Langetepe. On optimizing multi-sequence functionals for competitive analysis. In *Abstracts 21st European Workshop Comput. Geom.*, pages 111–114, 2005.

[13] T. Kamphans and E. Langetepe. Optimal competitive online ray search with an error-prone robot. In *Proc. 4th Internat. Workshop Efficient Experim. Algorithms*, volume 3503 of *Lecture Notes Comput. Sci.*, pages 593–596. Springer, 2005.

[14] M.-Y. Kao, J. H. Reif, and S. R. Tate. Searching in an unknown environment: An optimal randomized algorithm for the cow-path problem. *Inform. Comput.*, 133(1):63–79, 1996.

[15] E. Langetepe. *Design and Analysis of Strategies for Autonomous Systems in Motion Planning.* PhD thesis, Department of Computer Science, FernUniversität Hagen, 2000.

[16] E. Langetepe. On the optimality of spiral search. In *Proc. 21st Annu. ACM-SIAM Symp. Disc. Algor.*, 2010.

[17] E. Langetepe. Optimizing two-sequence functionals for competitive analysis. *Unpublished manuscript, submitted for publication*, 2010.

[18] A. López-Ortiz and S. Schuierer. The ultimate strategy to search on m rays? *Theor. Comput. Sci.*, 261(2):267–295, 2001.

[19] N. S. V. Rao, S. Kareti, W. Shi, and S. S. Iyengar. Robot navigation in unknown terrains: introductory survey of non-heuristic algorithms. Technical Report ORNL/TM-12410, Oak Ridge National Laboratory, 1993.

[20] S. Schuierer. Searching on m bounded rays optimally. Technical Report 112, Institut für Informatik, Universität Freiburg, Germany, 1998.

[21] S. Schuierer. Lower bounds in on-line geometric searching. *Comput. Geom. Theory Appl.*, 18:37–53, 2001.

[22] D. D. Sleator and R. E. Tarjan. Amortized efficiency of list update and paging rules. *Commun. ACM*, 28:202–208, 1985.